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Relationships between growth constants for animals and trees

D S Gaunt†, P J Peard†, C E Soteros‡ and S G Whittington§

† Department of Physics, King's College, Strand, London WC2R 2LS, UK

‡ Department of Mathematics and Statistics, University of Saskatchewan, Saskatoon, Canada S7N 0W0

§ Department of Chemistry, University of Toronto, Toronto, Canada M5S 1A1

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Abstract. We consider the growth constants of several types of animals on d-dimensional hypercubic lattices. A combination of rigorous inequalities and $1/d$-expansions leads us to conjecture a strict ordering of these growth constants. These results are useful in the analysis of models of branched polymer behaviour.

1. Introduction

Recently there has been considerable interest in using lattice animals (connected subgraphs of the lattice) and lattice trees (animals with no cycles) to model the collapse transition of branched polymers in dilute solution (Madras *et al* 1990, Flesia and Gaunt 1992, Flesia *et al* 1992, Stella *et al* 1992, Vanderzande 1993). Several models have been used involving different types of lattice animals. Each of these types grows exponentially with size and the growth constants are related to the free energies of the models at particular temperatures. The relative values of the growth constants are important because they determine the broad features of the temperature dependence of the free energies.

In this paper we shall be concerned with six types of lattice animal of which three are weakly embeddable (subgraphs of the lattice) and three are strongly embeddable (section graphs of the lattice) in a d-dimensional simple hypercubic lattice. Let a_n, b_n, t_n be the number per lattice site of animals with n vertices, animals with n edges and trees with n vertices, weakly embeddable in the lattice, respectively. Similarly we use A_n, B_n, T_n for the corresponding numbers of strongly embeddable objects. The smallest value of n for which these quantities are all different is $n = 4$ and, for the square lattice, $a_4 = 23$, $b_4 = 88$, $t_4 = 22$, $A_4 = 19$, $B_4 = 56$ and $T_4 = 18$.

A concatenation argument establishes the existence of the limit

$$\lim_{n\to\infty} n^{-1} \log a_n \equiv \log \lambda_s \tag{1.1}$$

and this defines the growth constant (λ_s) of weakly embeddable animals with n vertices, i.e. $a_n = \lambda_s^{n+o(n)}$. Similarly, the other constants are defined by the expressions $b_n = \lambda_b^{n+o(n)}$, $t_n = \lambda_o^{n+o(n)}$, $A_n = \Lambda_s^{n+o(n)}$, $B_n = \Lambda_b^{n+o(n)}$ and $T_n = \Lambda_o^{n+o(n)}$.

In this paper we give $1/d$-expansions for each of these six growth constants which strongly suggest the ordering

$$\lambda_s > \lambda_b > \lambda_o > \Lambda_s > \Lambda_b > \Lambda_o \, . \tag{1.2}$$

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Some of these can easily be proved to be *non-strict* inequalities ($\lambda_s \geqslant \lambda_o$, $\lambda_b \geqslant \lambda_o$, $\Lambda_s \geqslant \Lambda_o$, $\Lambda_b \geqslant \Lambda_o$, $\lambda_s \geqslant \Lambda_s$, $\lambda_b \geqslant \Lambda_b$, $\lambda_o \geqslant \Lambda_o$), by noticing that one set of animals is a subset of another set of animals. The strict inequalities are more difficult but Whittington and Soteros (1990) used pattern theorem arguments to show that

$$\lambda_s > \lambda_o > \Lambda_s > \Lambda_o \, .$$

Here we prove the additional strict inequalities $\lambda_s > \lambda_b > \lambda_o$.

Although these rigorous arguments establish that certain pairs of growth constants are different, they give no information about the magnitudes of these differences. These can be characterized by the power of $1/d$ at which the expansions of the two growth constants first differ and may be estimated numerically, in arbitrary d, by truncation.

2. Rigorous results

In this section we make extensive use of the arguments of Soteros and Whittington (1988) and Madras *et al* (1988) to prove that $\lambda_s > \lambda_b > \lambda_o$.

The cyclomatic index is the maximum number of edges which can be removed without disconnecting the animal. Let $a_n(c)$ be the number (per lattice site) of animals with n vertices, having cyclomatic index c. Similarly, let $b_n(c)$ be the number of animals with n edges and cyclomatic index c. Madras *et al* (1988) have shown that the limit exists in the following definition of $\phi(\alpha)$:

$$\log \phi(\alpha) = \lim_{n\to\infty} n^{-1} \log a_n(\lceil \alpha n \rceil) \tag{2.1}$$

and have investigated the properties of $\phi(\alpha)$.

Using Euler's relation we have

$$b_n(c) = a_{n-c+1}(c) \, . \tag{2.2}$$

This equation can be used to establish the existence of the limit in the definition of the function $\psi(\alpha)$,

$$\log \psi(\alpha) = \lim_{n\to\infty} n^{-1} \log b_n(\lceil \alpha n \rceil) \, . \tag{2.3}$$

This follows immediately from (2.1) and (2.2) and lemma 4.5 of Madras *et al* (1988) on setting $c = \lceil \alpha n \rceil$, taking logarithms, dividing by n, and letting n tend to infinity, giving

$$\log \psi(\alpha) = (1-\alpha) \log \phi \left(\frac{\alpha}{1-\alpha} \right) . \tag{2.4}$$

Next we prove that $\lambda_b > \lambda_o$. Clearly

$$\log \lambda_b \geqslant \log \psi(\alpha) = (1-\alpha) \log \phi \left(\frac{\alpha}{1-\alpha} \right) \tag{2.5}$$

for any $\alpha \leqslant 1 - 1/d$. To obtain a lower bound for the expression on the right-hand side of (2.5) we now make use of theorem 3 and (2.23) of Madras *et al* (1988). (In fact (2.23) of Madras *et al* (1988) relies on (2.21) of Soteros and Whittington (1988) being true for any dimension d. A detailed proof of (2.21) for $d = 2$ and a sketch for higher dimensions was given in Soteros and Whittington (1988); more details of a proof for $d > 2$ are given in the appendix.) This leads to the following result. Given a sufficiently small positive constant ϵ,

$$\log \lambda_b \geqslant (1-\alpha) \log \lambda_o + \epsilon(1-2\alpha) \log \left(\frac{\epsilon(1-2\alpha)}{\epsilon(1-2\alpha)-\alpha} \right) + \alpha \log \left(\frac{\epsilon(1-2\alpha)-\alpha}{3\alpha\lambda_o} \right) \tag{2.6}$$

for all positive α less than a given constant depending on ϵ. This equation can be rewritten as

$$\log\lambda_b \geqslant \log\lambda_o + \epsilon(1-2\alpha)\log\left(\frac{\epsilon(1-2\alpha)}{\epsilon(1-2\alpha)-\alpha}\right) + \alpha\log\left(\frac{\epsilon(1-2\alpha)-\alpha}{3\alpha\lambda_o^2}\right). \tag{2.7}$$

The second term on the right-hand side is always positive and the third term is positive if

$$0 < \alpha < \frac{\epsilon}{1+2\epsilon+3\lambda_o^2}. \tag{2.8}$$

Since it is always possible to choose a value of α to satisfy this condition we have

$$\lambda_b > \lambda_o. \tag{2.9}$$

To show that $\lambda_s > \lambda_b$ we use (2.4) which gives a relation between ψ and ϕ. Clearly $\lambda_s = \max_\alpha \phi(\alpha)$ and $\lambda_b = \max_\alpha \psi(\alpha)$. Suppose that $\psi(\alpha)$ first reaches its maximum value for $\alpha = \alpha_b$. Either $\alpha_b = 0$ or $\alpha_b > 0$. In the first case we have $\lambda_b = \psi(0) = \phi(0) = \lambda_o$ which is impossible because of (2.9). Hence

$$\begin{aligned}\log\lambda_b = \log\psi(\alpha_b) &= (1-\alpha_b)\log\phi\left(\frac{\alpha_b}{1-\alpha_b}\right)\\ &\leqslant (1-\alpha_b)\log\lambda_s < \log\lambda_s.\end{aligned} \tag{2.10}$$

Hence, we have proved that $\lambda_s > \lambda_b > \lambda_o$.

If we define α_s as the value of α at which $\phi(\alpha)$ first attains its maximum value, it is possible to derive an inequality relating α_s and α_b. By an argument analogous to that leading to (2.10) we can show that $\log\lambda_s \leqslant (1+\alpha_s)\log\lambda_b$. Together with (2.10) this implies that $(1+\alpha_s)(1-\alpha_b) \geqslant 1$ or, equivalently

$$\alpha_b \leqslant \frac{\alpha_s}{1+\alpha_s}. \tag{2.11}$$

3. 1/*d* expansions

In this section, we present expansions for the growth constants in inverse powers of $\sigma = 2d-1$. The methods used are similar to those described by Gaunt *et al* (1976, 1982), Gaunt and Ruskin (1978). Here we only give an outline of the argument for one of the cases.

We consider the number a_n of weakly embeddable animals counted by vertices. The first few terms can be written for general d as

$$\begin{aligned}
a_2 &= \binom{d}{1}\\
a_3 &= \binom{d}{1} + 4\binom{d}{2}\\
a_4 &= \binom{d}{1} + 21\binom{d}{2} + 32\binom{d}{3}\\
a_5 &= \binom{d}{1} + 93\binom{d}{2} + 444\binom{d}{3} + 400\binom{d}{4}\\
a_6 &= \binom{d}{1} + 418\binom{d}{2} + 4612\binom{d}{3} + 10\,944\binom{d}{4} + 6912\binom{d}{5}\\
&\vdots
\end{aligned} \tag{3.1}$$

These can be summarized as

$$a_n(d) = 2^{n-1}n^{n-3}\binom{d}{n-1} + 2^{n-3}n^{n-5}(n-2)(2n^2-2n-3)\binom{d}{n-2}$$
$$+2^{n-5}n^{n-7}\,\tfrac{1}{6}(n-3)(12n^5-56n^4+24n^3+41n^2-174n+1320)\binom{d}{n-3}$$
$$+2^{n-7}n^{n-9}\,\tfrac{1}{6}(n-4)(8n^8-80n^7+220n^6-114n^5-336n^4+3973n^3$$
$$-12\,749n^2+67\,226n-245\,280)\binom{d}{n-4}$$
$$+\cdots+\binom{d}{1} \tag{3.2}$$

for all values of n. Expanding the binomial coefficients in inverse power of σ, taking logarithms, dividing by n and letting $n \to \infty$, we obtain

$$\log\lambda_s(d) = \log\sigma + 1 - \tfrac{55}{24}\sigma^{-2} - \tfrac{53}{24}\sigma^{-3} - \cdots . \tag{3.3}$$

In a similar way we have derived the first few coefficients in the corresponding expansions for the other growth constants. Writing

$$\log\lambda = \log\sigma + 1 - \sum_{i\geqslant 1} c_i\sigma^{-i} \tag{3.4}$$

where λ is a generic growth constant, we give the values of the coefficients c_1, c_2, c_3 in table 1 for the six models. For λ_b, these agree with the results of Harris (1982) and for $\Lambda_s, \lambda_o, \Lambda_o$, c_1 and c_2 agree with the results of Gaunt *et al* (1976, 1982). The remaining coefficients are new.

A comparison of the coefficients in table 1 suggests $\lambda_s > \lambda_b > \lambda_o > \Lambda_s > \Lambda_b > \Lambda_o$ for sufficiently large d.

Table 1. Coefficients in $1/\sigma$-expansions of $\log\lambda$.

	c_1	c_2	c_3
$\log\lambda_s$	0	$\frac{55}{24}$	$\frac{53}{24}$
$\log\lambda_b$	$\frac{1}{2}$	$(\frac{8}{3} - 1/2e)$	$(\frac{85}{12} - 1/4e)$
$\log\lambda_o$	$\frac{1}{2}$	$\frac{8}{3}$	$\frac{85}{12}$
$\log\Lambda_s$	2	$\frac{79}{24}$	$\frac{317}{24}$
$\log\Lambda_b$	$\frac{5}{2}$	$(\frac{13}{6} - 1/2e)$	$(\frac{191}{12} + 1/4e)$
$\log\Lambda_o$	$\frac{5}{2}$	$\frac{13}{6}$	$\frac{191}{12}$

4. Discussion

The results of the last section suggest that (1.2) is satisfied for d sufficiently large. In addition, the results of section 2, coupled with those of Whittington and Soteros (1990), establish rigorously that $\lambda_s > \lambda_b > \lambda_o > \Lambda_s > \Lambda_o$. A pattern theorem argument similar to that of Whittington and Soteros can be used to show that $\lambda_b > \Lambda_b$ and $\Lambda_b > \Lambda_o$. This leaves open the inequality $\Lambda_s > \Lambda_b$. To address this, we reproduce (in table 2) numerical estimates of Λ_s and Λ_b, for $d = 2$ and 3, as given in Flesia and Gaunt (1992). These results are evidence for the strict inequality $\Lambda_s > \Lambda_b$ for the lowest dimensions.

Table 2. Series estimates of growth constants in low dimensions.

	$d = 2$	$d = 3$
Λ_s	4.063 ± 0.002	8.34 ± 0.025
Λ_b	3.877 ± 0.008	7.907 ± 0.004

Finally, it should be noted that the expansion defined by (3.4) is expected to be asymptotic rather than convergent (Kesten 1964, Fisher and Singh 1990, Hara and Slade 1994). Estimates of the various growth constants may then be obtained by truncation after the smallest term, although that is difficult to ascertain when only the first three coefficients are available. These estimates satisfy the inequalities in (1.2) for $d = 4, 5, 6, \ldots$. In almost all cases, the best estimate obtained by truncation is smaller than the result from exact enumeration and series analysis. The difference between the truncation estimate and the 'exact' result decreases as d increases and is essentially zero (to within the numerical uncertainties of the 'exact' results) by about $d = 4$–6.

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Appendix.

In this appendix we show that (2.21) of Soteros and Whittington (1988) (SW),

$$a_{n+c}(c) \geqslant A\binom{\epsilon n}{c} a_n(0)/3^c \tag{A.1}$$

holds in any dimension d. In SW, the inequality (A.1) was proved in detail for the square lattice and a sketch of a proof was given for higher dimensions. Here we give a revised and more detailed proof of (A.1) for $d > 2$ than that presented in SW. In particular, in the proof presented here the constants defined in (3.2) and (3.3) of SW will be independent of the dimension d. Just as for the square lattice case, we prove (A.1) using a sequence of theorems and lemmas. First, the notation introduced in SW must be generalized to d dimensions.

Let $(x_1, x_2, \ldots, x_d)$ be an arbitrary vertex in the hypercubic lattice Z^d. Given a tree T in Z^d with vertex set V, edge set E, n vertices and at least one vertex with degree greater than 2, let $\hat{V}$ be the set of all vertices in T with degree greater than 2. Consider $v_0 \in \hat{V}$ and let the coordinates of $v_0 = (y_1, y_2, \ldots, y_d)$. v_0 is contained in $\binom{d}{2}$ sub-planes of Z^d. The (j, k)-plane with $j < k$ is the plane $\{(x_1, x_2, \ldots, x_d)|x_i = y_i, i = 1, \ldots, d, i \neq j, k\}$. These planes can be ordered lexicographically according to the values of j and k. v_0's degree is at least three and thus v_0 has at least one pair of right-angled edges incident on it. Take the (j_0, k_0)-plane to be the first plane (in the lexicographic ordering of the planes) which contains at least one pair of right-angled edges incident on v_0. In the (j_0, k_0)-plane, the positive x_{k_0}-direction is considered north and the positive x_{j_0}-direction is considered east. In this plane, v_0 and the edges connected to it are in exactly one of the nine configurations

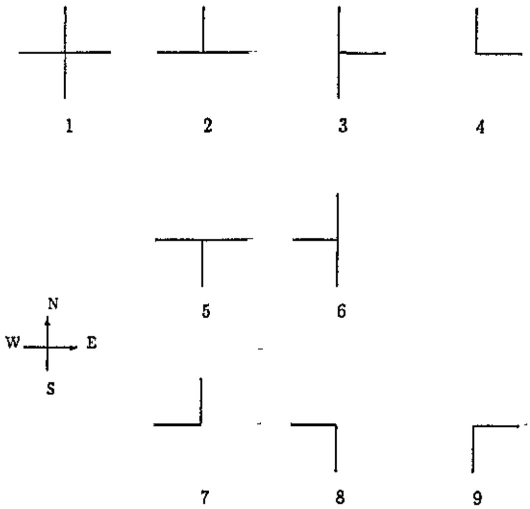

Figure A1. On the hypercubic lattice a vertex of degree greater than 2 must be one of the nine types shown.

shown in figure 1 with north and east defined as above. (Note that configurations 1, 2, 3, 5 and 6 correspond to the configurations 1–5, respectively, of SW.) v_0 is said to be a member of the set V_i if, looking in the (j_0, k_0)-plane, it is in the ith configuration of figure 1.

For any set S_0 of vertices in Z^d we define the top (bottom) vertex as follows. First construct the subset $S_1 \subset S_0$ such that the coordinate x_1 of every vertex in S_1 has the maximum (minimum) value over all vertices in S_0. We then recursively construct $S_k \subset S_{k-1}$ such that the coordinate x_k of every vertex in S_k has the maximum (minimum) value over all vertices in S_{k-1}. Let j be the smallest integer such that S_j contains precisely one vertex, and call this vertex the top (bottom) vertex of S_0.

Theorem A.1. Every tree (with n vertices) containing a vertex $v_0 \in V_1, V_2, V_3$ or V_4 can be converted into a 1-animal (with $n+1$ vertices) containing a 4-cycle in which v_0 is the bottom vertex of the 4-cycle. The resulting 1-animal can have at most three trees rooted at a vertex in $V_1 \cup V_2 \cup V_3 \cup V_4$ as precursors.

Proof. Let v_t be the top vertex of the tree, with coordinates $(x_1^t, x_2^t, \ldots, x_d^t)$. In the following we assume for convenience that v_0 is such that $j_0 = 1$ and $k_0 = 2$. (To obtain the required construction for other v_0, rotate the tree so that 'east' is in the positive x_1-direction and 'north' is in the positive x_2-direction, perform the construction and then rotate back.)

Since $v_0 \in V_1, V_2, V_3$ or V_4 then v_0 is connected to v_1 and v_2 with coordinates $(y_1+1, y_2, \ldots, y_d)$ and $(y_1, y_2+1, y_3, \ldots, y_d)$ respectively. We consider four subcases according to whether

(i) there is no vertex in the tree with coordinates $(y_1+1, y_2+1, y_3, \ldots, y_d)$, (in this case we say $v_0 \in W_1$),
(ii) there is a vertex v_3 in the tree with coordinates $(y_1+1, y_2+1, y_3, \ldots, y_d)$ and either $(v_1 - v_3) \in E$ or $(v_2 - v_3) \in E$, (then we say $v_0 \in W_2$),
(iii) $v_3 \in V$, $(v_1 - v_3) \notin E$, $(v_2 - v_3) \notin E$ and, using the definition of k^* given below, either $k^* = 5$ or k^* is even (in this case we say $v_0 \in W_3$), or
(iv) $v_3 \in V$, $(v_1 - v_3) \notin E$, $(v_2 - v_3) \notin E$ and where $k^* > 5$ and odd (then we say $v_0 \in W_4$).

Since T is a tree it is not possible for both $(v_1-v_3) \in E$ and $(v_2-v_3) \in E$. For cases (iii) and (iv), the tree must contain at least one of the $2d-2$ vertices: $v_4 = (y_1+2, y_2+1, y_3, \ldots, y_d)$, $v_5 = (y_1+1, y_2+2, y_3, \ldots, y_d)$, $v_6 = (y_1+1, y_2+1, y_3+1, y_4, \ldots, y_d)$, $v_7 = (y_1+1, y_2+1, y_3-1, y_4, \ldots, y_d), \ldots,$ $v_{2d} = (y_1+1, y_2+1, y_3, \ldots, y_{d-1}, y_d+1)$, $v_{2d+1} = (y_1+1, y_2+1, y_3, \ldots, y_{d-1}, y_d-1)$. v_3 is then connected to v_0 by a path containing an edge, $(v_3 - v_k)$, through one and only one of the vertices $v_k, k = 4, \ldots, 2d+1$; define $k^* \in \{4, 5, 6, \ldots, 2d+1\}$ to be the subscript such that $v_{k^*} \in V$ is this vertex.

For the four cases we have four different constructions (the first two are exactly the same as in the $d = 2$ case):

(i) add v_3 at $(y_1+1, y_2+1, y_3, \ldots, y_d)$ and the edges $(v_1 - v_3)$ and $(v_2 - v_3)$;
(ii) if $(v_1-v_3) \in E$, add (v_2-v_3), and the vertex $v_{t'}$ with coordinates $\left(x_1^t+1, x_2^t, x_3^t, \ldots, x_d^t\right)$ and the edge $(v_t-v_{t'})$. If $(v_2-v_3) \in E$, add (v_1-v_3), and the vertex $v_{t''}$ with coordinates $\left(x_1^t, x_2^t+1, x_3^t, \ldots, x_d^t\right)$ and the edge $(v_t - v_{t''})$;
(iii) delete the edge $(v_3 - v_{k^*})$ and add the edges $(v_1 - v_3)$ and $(v_2 - v_3)$. Then, if $k^* = 4$ add the vertex $v_{t'} = \left(x_1^t+1, x_2^t, \ldots, x_d^t\right)$ and edge $(v_t - v_{t'})$. If $k^* = 5$, add the vertex $v_{t''} = \left(x_1^t, x_2^t+1, x_3^t, \ldots, x_d^t\right)$ and edge $(v_t - v_{t''})$. Finally, if $k^* = 2j, j > 2$, add the vertex $v_{tj} = \left(x_1^t, x_2^t, x_3^t, \ldots, x_j^t+1, \ldots, x_d^t\right)$ and edge $(v_t - v_{tj})$;
(iv) delete the edge $(v_3 - v_{k^*})$ and add the edges $(v_1 - v_3)$ and $(v_2 - v_3)$. Then add the vertex $v_{tj} = \left(x_1^t, x_2^t, x_3^t, \ldots, x_j^t+1, \ldots, x_d^t\right)$ and edge $(v_t - v_{tj})$ where $j = (k^*-1)/2$ (note that $j > 2$).

The connected graph resulting from each of these constructions has $n+1$ vertices and $n+1$ edges so that it is a 1-animal. Case (ii) 1-animals can be distinguished from case (iv) 1-animals by looking at the direction of the edge attached to the top vertex; hence these two cases can be combined.

Let $\mathcal{T}$ be the set of trees such that $T \in \mathcal{T}$ iff $V_1(T) \cup V_2(T) \cup V_3(T) \cup V_4(T)$ is not empty. Let $\mathcal{T}_R$ be the set of rooted trees obtained by rooting each member (T) of $\mathcal{T}$ at each vertex $v_0 \in V_1(T) \cup V_2(T) \cup V_3(T) \cup V_4(T)$. Let $\mathcal{T}_{R_k} \subset \mathcal{T}_R$ be such that the tree $T \in \mathcal{T}_R$ is a member of $\mathcal{T}_{R_k}$ iff $v_0(T) \in W_k(T)$.

The transformation defined above maps a member of $\mathcal{T}_{R_k}$ uniquely into a 1-animal so that this transformation from $\mathcal{T}_{R_k}$ is 1–1 and onto the image set of $\mathcal{T}_{R_k}$. Furthermore, the transformation maps a member of $\mathcal{T}_{R_2} \cup \mathcal{T}_{R_4}$ uniquely into a 1-animal so that this transformation is 1–1 and onto the image set of $\mathcal{T}_{R_2} \cup \mathcal{T}_{R_4}$. Hence, each 1-animal can have at most three precursors in the set of rooted trees. This completes the proof. □

Let $b_n(\epsilon)$ be the number of trees with n vertices, more than ϵn of which are members of $V_1 \cup V_2 \cup V_3 \cup V_4$. If a tree has more than ϵn vertices in $V_1 \cup V_2 \cup V_3 \cup V_4$, c vertices can be chosen in at least $\binom{\epsilon n}{c}$ ways and hence, using an argument analogous to that in SW,

$$a_{n+c}(c) \geqslant \binom{\epsilon n}{c} b_n(\epsilon)/3^c \tag{A.2}$$

for $\epsilon n \geqslant c$.

The result corresponding to lemma 1 in SW is now as follows.

Lemma A.1. If $t_n(\epsilon, >)$ is the number of trees with n vertices containing more than ϵn vertices of degree greater than 2 then

$$b_n(\epsilon/9) \geqslant t_n(\epsilon, >)/4. \tag{A.3}$$

Proof. Suppose that $S_n(\epsilon, >)$ is the set of trees with n vertices having more than ϵn vertices of degree greater than 2. We construct subsets $S_{nm}(\epsilon, >)$ such that a tree $T \in S_n(\epsilon, >)$ is a member of $S_{nm}(\epsilon, >)$ if the number of vertices in $V_m(T)$ is at least as large as the number in $V_i(T), i = 1, \ldots, 9, i \neq m$ and m is the smallest value for which this is true. Thus T can be a member of only one subset $S_{nm}(\epsilon, >)$. Using the symmetry between V_2, V_3, V_5 and V_6, $|S_{n2}(\epsilon, >)| \geqslant |S_{n3}(\epsilon, >)| \geqslant |S_{n5}(\epsilon, >)| \geqslant |S_{n6}(\epsilon, >)|$ where we write $|\cdots|$ for the cardinality of a set. Using the symmetry between V_4, V_7, V_8 and V_9, $|S_{n4}(\epsilon, >)| \geqslant |S_{n7}(\epsilon, >)| \geqslant |S_{n8}(\epsilon, >)| \geqslant |S_{n9}(\epsilon, >)|$. Hence

$$\sum_{k=1}^{4} |S_{nk}(\epsilon, >)| \geqslant |S_n(\epsilon, >)| /4 = t_n(\epsilon, >)/4 . \tag{A.4}$$

Any $T \in S_{nm}(\epsilon, >)$ is also a member of $S_n(\epsilon, >)$ and hence has at least $n\epsilon/9$ vertices in $V_m(T)$. Therefore the number of trees having at least $n\epsilon/9$ vertices in $V_1 \cup V_2 \cup V_3 \cup V_4$ is bounded below by $\sum_{k=1}^{4} |S_{nk}(\epsilon, >)|$ and this together with (A.4), implies (A.3). □

The next two lemmas have already been proved for trees in Z^d for arbitrary dimension d.

Lemma A.2 (Lipson and Whittington 1983). If $t_n(\epsilon, \leqslant)$ is the number of trees with n vertices containing at most ϵn vertices of degree greater than 2 then there exists a positive constant $\lambda(\epsilon)$ *such that the limit*

$$\lim_{n\to\infty} n^{-1} \log t_n(\epsilon, \leqslant) \equiv \log \lambda(\epsilon) < \infty \tag{A.5}$$

exists.

Lemma A.3 (Soteros and Whittington 1988). $\lambda(\epsilon)$ is a log concave function of ϵ on [0,1].

The following lemma was proved in SW for $d = 2$ and the proof is easily modified to work for arbitrary dimension d.

Lemma A.4. $\log \lambda(\epsilon)$ is a continuous function of ϵ in [0,1].

Proof. The steps of the proof are exactly the same as those described in SW except that now instead of their equations (2.8) and (2.9) we obtain

$$t_n(\epsilon, \leqslant) \leqslant u_n(2d\epsilon) \tag{A.6}$$

since

$$m = n_1 + n_3 + n_4 + \cdots + n_{2d} = 2 + 2n_3 + 3n_4 + \cdots + (2d-1)n_{2d} \leqslant 2d\epsilon n \tag{A.7}$$

provided that $2/n \leqslant \epsilon \leqslant 1/(2d)$. Making appropriate changes in the remainder of the proof but following the same steps leads to the desired result. □

Just as in SW these lemmas lead to the following results.

Lemma A.5. There exists $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$,

$$\lim_{n\to\infty} (t_n(\epsilon, >)/a_n(0)) = 1 . \tag{A.8}$$

Lemma A.6. There exists an $A > 0$ and an integer N such that for all $n > N$

$$b_n(\epsilon) \geqslant Aa_n(0) \tag{A.9}$$

for any $\epsilon \leqslant \epsilon_0/9$.

It then follows immediately from (A.2) and (A.9) that there exists $\epsilon_0 > 0$, $A > 0$ and an integer N such that for any $\epsilon \leqslant \epsilon_0/9$ (A.1) holds for all $n > N$.

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